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AN APPLICATION OF QUANTITATIVE SUBSPACE THEOREM

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1. Introduction

We give here one kind of generalization of the Schmidt-Schlickewei quantitative subspace theorem, using the argument of M. Ru and P. -M. Wong. We consider linear forms not only of the same number as variables but also of a larger number than the number of variables.

Let K be a normal extension of \mathbf{Q} of degree d over \mathbf{Q} . Let $M(K)$ be the set of non-equivalent places of K : for $v \in M(K)$, denote by $|\cdot|_v$ the corresponding absolute value normalized such that for $a \in \mathbf{Q}$, $|a|_v = |a|$ if v lies above the archimedean prime of \mathbf{Q} , and $|p|_v = 1/p$ if v lies above the rational prime p . Let $M_\infty(K)$ be the set of archimedean places of K .

We consider $S \subset M(K)$ (not necessarily containing $M_\infty(K)$), of cardinality $s < \infty$. Let K_v be the completion of K with respect to $|\cdot|_v$. Put $d_v = [K_v : \mathbf{Q}_v]$ for the local degree. For $a \in K$, we put $\|a\| = |a|_v^{\frac{d_v}{d}}$.

For $\mathbf{a} = (a_0, \dots, a_n) \in K^{n+1}$, $v \in M(K)$, write

$$|\mathbf{a}|_v = (|a_0|_v^2 + \dots + |a_n|_v^2)^{\frac{1}{2}}$$

if v is archimedean, and

$$|\mathbf{a}|_v = \max_{0 \leq i \leq n} |a_i|_v$$

if v is nonarchimedean. Let us denote $\|\mathbf{a}\|_v = |\mathbf{a}|_v^{\frac{d_v}{d}}$. We define the height of \mathbf{a} by

$$H(\mathbf{a}) = \prod_{v \in M(K)} \|\mathbf{a}\|_v$$

and write $h(\mathbf{a}) = \log H(\mathbf{a})$.

It is well-known that the definition of $H(\mathbf{a})$ is independent of a choice of field where \mathbf{a} lies, and also that $H(c\mathbf{a}) = H(\mathbf{a})$ for $c \in K - \{0\}$. Given a linear form $L(\mathbf{x}) = a_0x_0 + \dots + a_nx_n$

with coefficients $a_0, \dots, a_n \in K$ not all zero, we put $H(L) = H(\mathbf{a})$ for $\mathbf{a} = (a_0 \dots a_n)$ as the height of L . For $v \in M(K)$ we denote $\|L\|_v = \|\mathbf{a}\|_v$.

2. Quantitative subspace theorem

A higher dimensional case of Roth's theorem, that we call the subspace theorem, is established by W. M. Schmidt for archimedean places. A quantitative version is also derived by himself, and extended by H. P. Schlickewei to nonarchimedean places. See for the historical survey in [Schl] [Schm 1] [Schm 2]. We apply here the theorem of Schlickewei in [Schl] which is stated as follows.

Theorem 2.1 (Schlickewei).

Let K, S as above. Suppose that for each $v \in S$ we are given $n+1$ linearly independent linear forms $L_1^{(v)}, \dots, L_{n+1}^{(v)}$ in $n+1$ variables with coefficients in K . Let $0 < \delta < 1$. Consider the inequality

$$\prod_{v \in S} \prod_{i=1}^{n+1} \frac{\|L_i^{(v)}(\mathbf{x})\|_v}{\|L_i^{(v)}\|_v \|\mathbf{x}\|_v} < H(\mathbf{x})^{-n-1-\delta}.$$

Then there exists proper subspaces S_1, \dots, S_{t_1} of K^{n+1} with

$$t_1 = \lceil (8sd)^{2^{34}(n+1)d} s^6 \delta^{-2} \rceil$$

such that every solution $\mathbf{x} \in K^{n+1}$ lies in

$$\bigcup_{i=1}^{t_1} S_i \cup D$$

where

$$D = \{\mathbf{x} \in K^{n+1} \ ; \ H(\mathbf{x}) < \max((n+1)!^{\frac{9}{\delta}}, H(L_i^{(v)})^{\frac{9d(n+1)s}{\delta}} (v \in S, i = 1, \dots, n+1))\}.$$

3. Preliminaries

We recall here the definition of subgeneral position and Nochka weight following [R-W]. Let $1 \leq k \leq n < q$ be rational integers. Consider nonzero distinct q linear forms in $k+1$ variables with coefficients in K . For each linear form $L_i(\mathbf{x}) = a_{i0}x_0 + \dots + a_{ik}x_k$, put $\mathbf{a}_i = \mathbf{a}(L_i) = (a_{i0}, \dots, a_{ik}) \in K^{k+1}$ ($1 \leq i \leq q$). The linear forms L_1, \dots, L_q are called in n -subgeneral position if any distinct $n+1$ elements of the set $\{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ span K^{k+1} . We see that n -subgeneral position is equivalent to general position when $n = k$.

Now we define Nochka weight (cf [R-W]).

Let $1 \leq k \leq n < q$ be rational integers and L_1, \dots, L_q be linear forms in $k+1$ variables with coefficients in K , supposed to be in n -subgeneral position. We denote the dimension of

the linear span over K of a subset $B \subset A := \{\mathbf{a}_1, \dots, \mathbf{a}_q\}$ by $d(B)$. Put $P(B) = (\#B, d(B))$ which is regarded as a point in \mathbf{R}^2 . For two points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ in \mathbf{R}^2 with $x_1 \neq x_2$, we write $\sigma(P_1, P_2) = \frac{y_1 - y_2}{x_1 - x_2}$. Proposition 2.1 in [R-W] (under some corrections) allows us to show that there exists a sequence of subsets

$$A = B_{s+1} \supset B_s \supset B_{s-1} \cdots \supset B_1 \supset B_0 = \emptyset$$

where the sequence of numbers $\sigma(P(B_{i+1}), P(B_i))$ ($0 \leq i \leq s$) is uniquely determined.

If an element $\mathbf{a} \in A$ lies in $B_{i+1} - B_i$ ($0 \leq i \leq s$), we put $\omega(\mathbf{a}) = \sigma(P(B_{i+1}), P(B_i))$, which is called Nochka weight. For simplicity, we write $\sigma(P(B_{i+1}), P(B_i)) = \sigma_i$. Several properties of Nochka weight are presented in [R-W].

4. Results

For simplicity, we restrict here $K \subset \mathbf{R}$ and consider $S = \{\infty\}$; one archimedean place of K defined by $|x|_\infty = \max(x, -x)$. Put $|x|_\infty = |x|$.

For $1 \leq k \leq n < q$, consider linear forms L_1, \dots, L_q in $k+1$ variables with coefficients in K , supposed to be in n -subgeneral position. Write $\mathbf{a}_i = (a_{i0}, \dots, a_{ik})$ a coefficient vector of L_i respectively ($1 \leq i \leq q$). Then for all $\mathbf{x} \in \mathbf{R}^{k+1}$ we claim

$$\# \left\{ i; \frac{\|L_i(\mathbf{x})\|}{\|L_i\| \|\mathbf{x}\|} < c_0 \right\} \leq n$$

with

$$c_0 = \frac{1}{2} \min_{\mathbf{a}_i \neq \mathbf{a}_j} \left(1 - \frac{|(\mathbf{a}_i, \mathbf{a}_j)|}{\|\mathbf{a}_i\| \|\mathbf{a}_j\|} \right)^{\frac{d}{2}}$$

where $(\mathbf{a}_i, \mathbf{a}_j) = a_{i0}a_{j0} + \dots + a_{ik}a_{jk}$.

Using Theorem 2.1, we get the following quantitative statement of Theorem 3.3 of [R-W].

Theorem 4.1.

Let K, S as above.

Let $1 \leq k \leq n < q$ be rational integers and L_1, \dots, L_q be linear forms in $k+1$ variables with coefficients in K , supposed to be in n -subgeneral position. Let $\omega_i = \omega(\mathbf{a}_i)$ be the associated Nochka weight with L_i ($1 \leq i \leq q$). Let $0 < \delta < 1$. Consider the inequality

$$\sum_{i=1}^q \omega_i \log \left(\frac{\|L_i\| \|\mathbf{x}\|}{\|L_i(\mathbf{x})\|} \right) > (k+1+\delta) \log |\mathbf{x}|.$$

Then there exists proper subspaces S_1, \dots, S_{t_2} of K^{k+1} with

$$t_2 = \lceil 32d^{2^{34}(k+1)d\delta^{-2}} \rceil$$

such that every solution $\mathbf{x} \in \mathbf{Z}^{k+1}$ with $L_i(\mathbf{x}) \neq 0$ for all $1 \leq i \leq q$ lies in

$$\bigcup_{i=1}^{t_2} S_i \cup D_1 \cup D_2$$

where

$$D_1 = \left\{ \mathbf{x} \in \mathbf{Z}^{k+1} \ ; \ |\mathbf{x}| < \exp\left(\frac{2c_1}{\delta}\right) \right\},$$

$$D_2 = \left\{ \mathbf{x} \in \mathbf{Z}^{k+1} \ ; \ H(\mathbf{x}) < \max\left((k+1)!^{\frac{18}{\delta}}, H(L_i)^{\frac{18d(k+1)}{\delta}}\right) \right\}$$

and

$$c_1 = \frac{(q-n)(k+1)}{n+1} \log \frac{1}{c_0}.$$

Outline of the proof of Theorem 4.1

We follow the argument of Ru-Wong. Take c_0 as above. For $1 \leq i \leq q$, put $E_i(\mathbf{x}) = \frac{\|L_i\| \|\mathbf{x}\|}{\|L_i(\mathbf{x})\|}$. Then we have $\#I(\mathbf{x}) \leq n$ where $I(\mathbf{x}) = \left\{ i; \log E_i(\mathbf{x}) \geq \log \frac{1}{c_0} \right\}$. Lemma 3.1 of [R-W] implies that there exists a set $J(\mathbf{x})$ of cardinality $k+1$ such that $\{\mathbf{a}_i; i \in J(\mathbf{x})\}$ are linearly independent and

$$\prod_{i \in I(\mathbf{x})} E_i(\mathbf{x})^{\omega_i} \leq \prod_{i \in J(\mathbf{x})} E_i(\mathbf{x})$$

with $\omega_i = \omega(\mathbf{a}_i)$ for $L_i(\mathbf{x}) = (\mathbf{a}_i, \mathbf{x})$. Therefore

$$\prod_{i \in J(\mathbf{x})} E_i(\mathbf{x}) \leq \max_I \prod_{i \in I} E_i(\mathbf{x})$$

where I runs over the family of all subsets of $\{1, \dots, q\}$ with $\#I = k+1$ and $\{\mathbf{a}_i; i \in I\}$ linearly independent. Using the property $\omega_i \leq \frac{k+1}{n+1}$ of Nochka weight, we obtain

$$\sum_{i=1}^q \omega_i \log E_i(\mathbf{x}) = \sum_{i \in I(\mathbf{x})} \omega_i \log E_i(\mathbf{x}) + c_1 \leq \max_I \sum_{i \in I} \log E_i(\mathbf{x}) + c_1$$

with $c_1 = \frac{(q-n)(k+1)}{n+1} \log \frac{1}{c_0}$. Thus the solutions $\mathbf{x} \in \mathbf{Z}^{k+1}$ outside of L_1, \dots, L_q of the inequality

$$\sum_{i=1}^q \omega_i \log E_i(\mathbf{x}) > (k+1+\delta) \log |\mathbf{x}|$$

are contained in the solutions $\mathbf{x} \in \mathbf{Z}^{k+1}$ outside of L_1, \dots, L_q of the inequality

$$(4.2) \quad \max \sum_{i \in I}^q \log E_i(\mathbf{x}) > (k+1+\delta) \log |\mathbf{x}| - c_1.$$

Then the solutions of (4.2) are contained in the union of the set D_1 and the set of the solutions of

$$(4.3) \quad \max_{i \in I} \sum_{i \in I}^q \log E_i(\mathbf{x}) > (k+1 + \frac{\delta}{2}) \log |\mathbf{x}|,$$

because the solutions of (4.2) with $\log |\mathbf{x}| \geq \frac{2c_1}{\delta}$ satisfies

$$(k+1 + \delta) \log |\mathbf{x}| - c_1 \geq (k+1 + \frac{\delta}{2}) \log |\mathbf{x}|.$$

Now we apply Theorem 2.1 for $k+1$ variables to solve (4.3) which establishes our statement.

Applying this theorem, we obtain a quantitative statement of Theorem 3.5 of [R-W] as follows.

Theorem 4.4.

Let $1 \leq k \leq n < q$ with $q > 2n - k + 1$ be rational integers and L_1, \dots, L_q be linear forms in $k+1$ variables with coefficients in K , in n -subgeneral position.

Let $0 < \delta < 1$. Consider the inequality

$$\sum_{i=1}^q \log \frac{\|L_i\| \|\mathbf{x}\|}{\|L_i(\mathbf{x})\|} > (2n - k + 1 + \delta) \log |\mathbf{x}|.$$

Then there exists proper subspaces S_1, \dots, S_{t_3} of K^{k+1} with

$$t_3 = \lceil 128d^{2^{34(k+1)d}(2n+1)^2\delta^{-2}} \rceil$$

such that every solution $\mathbf{x} \in \mathbf{Z}^{k+1}$ with $L_i(\mathbf{x}) \neq 0$ for all $1 \leq i \leq q$ lies in

$$\bigcup_{i=1}^{t_3} S_i \bigcup D_3 \bigcup D_4$$

where

$$D_3 = \left\{ \mathbf{x} \in \mathbf{Z}^{k+1} \ ; \ |\mathbf{x}| < \exp \left(\frac{4(2n+1)c_1}{\delta} \right) \right\},$$

$$D_4 = \left\{ \mathbf{x} \in \mathbf{Z}^{k+1} \ ; \ H(\mathbf{x}) < \max \left((k+1)!^{\frac{36(2n+1)}{\delta}}, \right. \right. \\ \left. \left. H(L_i)^{\frac{36d(2n+1)(k+1)}{\delta}} \ (i = 1, \dots, q) \right) \right\}$$

with c_1 in Theorem 4.1.

Outline of the proof of Theorem 4.4

Put $\theta = \frac{1}{\sigma_s}$. Then $\theta = \frac{q-2n+k-1}{\omega_1+\dots+\omega_q-(k+1)}$ and $\frac{n+1}{k+1} \leq \theta \leq \frac{2n-k+1}{k+1} \leq 2n+1$ by

Theorem 4.3 (1) (3) of [R-W]. For all $v \in M(K)$ and for \mathbf{x} outside of zeroes of L_i , we claim that

$$\log \frac{\|L_i\|_v \|\mathbf{x}\|_v}{\|L_i(\mathbf{x})\|_v} > 0$$

which derives

$$\log E_i(\mathbf{x}) \leq \sum_{v \in M(K)} \log \frac{\|L_i\|_v \|\mathbf{x}\|_v}{\|L_i(\mathbf{x})\|_v} = h(\mathbf{a}_i) + h(\mathbf{x}),$$

because we see $\sum_{v \in M(K)} \log \|L_i(\mathbf{x})\|_v = 0$ by the product formula. For $\varepsilon_1 > 0$, we get that a point \mathbf{x} either is contained in D_5 or satisfies $\log E_i(\mathbf{x}) \leq (1 + \varepsilon_1)h(\mathbf{x})$ where

$$D_5 = \left\{ \mathbf{x} \ ; \ h(\mathbf{x}) < \frac{\max_{1 \leq i \leq q} h(\mathbf{a}_i)}{\varepsilon_1} \right\}.$$

Therefore for $\mathbf{x} \notin D_5$, we have

$$\begin{aligned} & \sum_{1 \leq i \leq q} \log E_i(\mathbf{x}) \\ &= \sum_{1 \leq i \leq q} (1 - \theta \omega_i) \log E_i(\mathbf{x}) + \theta \sum_{1 \leq i \leq q} \omega_i \log E_i(\mathbf{x}) \\ &\leq (1 + \varepsilon_1) h(\mathbf{x}) \sum_{1 \leq i \leq q} (1 - \theta \omega_i) + \theta \sum_{1 \leq i \leq q} \omega_i \log E_i(\mathbf{x}). \end{aligned}$$

Consider $\mathbf{x} \in \mathbf{Z}^{k+1}$ with $L_i(\mathbf{x}) \neq 0$. By theorem 4.1, the inequality

$$\begin{aligned} \sum_{1 \leq i \leq q} \log E_i(\mathbf{x}) &\leq (1 + \varepsilon_1) h(\mathbf{x}) \left(q - \theta \sum_{1 \leq i \leq q} \omega_i \right) \\ &\quad + \theta (k + 1 + \delta_1) \log |\mathbf{x}| \end{aligned}$$

holds for all

$$\mathbf{x} \notin \bigcup_{i=1}^{t_2} S_i \cup D_1 \cup D_2 \cup D_5,$$

with $\delta = \delta_1$ in theorem 4.1. Since we have $h(\mathbf{x}) \leq \log |\mathbf{x}|$ for $\mathbf{x} \in \mathbf{Z}^{k+1}$, using the property of θ mentioned above, \mathbf{x} satisfies

$$\begin{aligned}
\sum_{1 \leq i \leq q} \log E_i(\mathbf{x}) &\leq (1 + \varepsilon_1)(2n - k + 1 - \theta(k + 1)) \log |\mathbf{x}| \\
&\quad + \theta(k + 1 + \delta_1) \log |\mathbf{x}| \\
&= (2n - k + 1 + \delta_1 \theta + \varepsilon_1(2n - k + 1 - \theta(k + 1))) \log |\mathbf{x}|.
\end{aligned}$$

For any $0 < \delta < 1$, take $\delta_1 = \varepsilon_1 = \frac{\delta}{2(2n+1)}$. Then $\varepsilon_1 \leq \frac{\delta}{2(2n-k+1-\theta(k+1))}$ if $2n - k + 1 - \theta(k + 1) \neq 0$, and otherwise we have $\varepsilon_1(2n - k + 1 - \theta(k + 1)) = 0$. This implies $\delta \geq \delta_1 \theta + \varepsilon_1(2n - k + 1 - \theta(k + 1))$ which shows that the solutions $\mathbf{x} \in \mathbf{Z}^{k+1}$ with $L_i(\mathbf{x}) \neq 0$ for all $1 \leq i \leq q$ of the inequality in the statement of the Theorem lie in the desired region.

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